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# ON FIRST INTEGRALS OF A NONHOLONOMIC MECHANICAL SYSTEM

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We consider a nonholonomic mechanical system with kinetic energy $T = 1/2\ g_{\lambda\mu}\dot{q}^{\lambda}\dot{q}^{\mu}$ subjected to nonholonomic constraints $\omega_{\varkappa}{}^{p}\dot{q}^{\varkappa} = 0$, and a corresponding constraint-free system [1]. The indices $\lambda, \mu, \nu, \ldots$ assume the values $1, 2, \ldots, n$; $a, b, c, \ldots$ are $1, 2, \ldots, k$ and $p, q, r, \ldots$ are $k, \ldots, n$. The equation of the constraint-free system have the form

$$\delta\dot{q}^{\varkappa}/dt = Q^{\varkappa}, \qquad \delta\dot{q}^{\varkappa}/dt = \ddot{q}^{\varkappa} + \Gamma_{\lambda\mu}^{\varkappa}\dot{q}^{\lambda}\dot{q}^{\mu}, \qquad Q^{\varkappa} = g^{\varkappa\nu}\partial U/\partial q^{\nu} \tag{1}$$

$$\Gamma_{\lambda\mu}^{\varkappa} = \frac{1}{2}g^{\varkappa\nu}\left(\frac{\partial g_{\nu\lambda}}{\partial q^{\mu}} + \frac{\partial g_{\nu\mu}}{\partial q^{\lambda}} - \frac{\partial g_{\lambda\mu}}{\partial q^{\nu}}\right)$$

while the equations of the nonholonomic system are

$$Ds^{-a}/dt = F^{a} \tag{2}$$

and the expressions for $Ds^{-a}/dt$, $\Gamma^{a}_{bc}$ and $F^{a}$ are given in [1].

The conditions for the expression $\xi_{\varkappa}\dot{q}^{\varkappa} = C$ to be a linear integral of a nonholonomic system are [1 – 4]

$$\Delta_c\lambda_a + \Delta_a\lambda_c = 0, \qquad \lambda_c F^c = 0 \tag{3}$$

$$\left(\lambda_a = \xi_{\varkappa}\alpha_a{}^{n}, \quad \nabla_c\lambda_a = \frac{\partial\lambda_a}{\partial q_{\varkappa}}\alpha_c{}^{n} - \Gamma_{ca}^{b}\lambda_b\right)$$

The necessary and sufficient condition for a nonholonomic system [5] to have the first integral $\xi_{\varkappa}\dot{q}^{\varkappa} = C$ linear in the Lagrangian velocities is, that this integral is a linear integral of the geodesics of the metric space $V_n$ with the metric tensor $g_{\lambda\mu}$ and, that the vectors $Q^{\varkappa}$ and $\omega_{\varkappa}{}^{p}$ are orthogonal to the vector $\xi_{\varkappa}$, i.e.

$$\nabla_{\varkappa}\xi_{\nu} + \nabla_{\nu}\xi_{\varkappa} = 0, \quad \omega_{\varkappa}{}^{p}\xi^{\varkappa} = 0, \quad Q_{\varkappa}\xi^{\varkappa} = 0 \tag{4}$$

Both groups of conditions (3) and (4) differ from each other. Let us inspect this feature in greater detail. Let $\xi_{\varkappa}\dot{q}^{\varkappa} = C$ be a linear integral of a nonholonomic system. Let $\eta_{\varkappa} = \xi_{\varkappa} + \rho_p w^{p}{}_{\varkappa}$, where $\rho_p$ are functions of the generalized coordinates $q^{\varkappa}$. Then, using the equations of constraints, we obtain

$$\eta_{\varkappa}\dot{q}^{\varkappa} = (\xi_{\varkappa} + \rho_p\omega_{\varkappa}{}^{p})\,\dot{q}^{\varkappa} = \xi_{\varkappa}\dot{q}^{\varkappa}$$

Consequently, $\eta_{\varkappa}\dot{q}^{\varkappa} = C$ represents the same integral of the nonholonomic system written in a different form. We shall say that the vectors $\eta_{\varkappa}$ all generate the same linear integral. Let us expand the vector $\xi_{\varkappa}$ in terms of the vectors $\omega^{\nu}$ [1]. We have $\xi_{\varkappa} = \rho_a\omega_{\varkappa}{}^{a} + \rho_p\omega_{\varkappa}{}^{p}$ and this implies that $\eta_{\varkappa} = \rho_a\omega_{\varkappa}{}^{a}$ generates the same linear integral.

The vector $\eta_{\varkappa} = \rho_a\omega_{\varkappa}{}^{a}$ is unique amongst the vectors generating the given linear integral and satisfies the second condition of (4), i.e. it lies in the admissible space. Consequently the above condition is not a necessary one for the existence of the linear integral; as we said in [1], it is only sufficient.

Let $\eta_{\varkappa} = \rho_a\omega_{\varkappa}{}^{a}$ and see how it affects the conditions (3). Using the formula [1]

$$g^{\lambda\mu} = G^{ab}\alpha_a{}^{\lambda}\alpha_b{}^{\mu} + G^{pq}\alpha_p{}^{\lambda}\alpha_q{}^{\mu}$$

and taking into account the fact that $\omega_x{}^a\alpha_p{}^x = 0$ [1], we transform the left-hand side of the second condition of (3) into

$$\lambda_c F^c = \eta_x \alpha_c{}^x G^{cb} Q_\nu \alpha_b{}^\nu = \eta_x Q_\nu (g^{x\nu} - G^{pq}\alpha_p{}^x\alpha_q{}^\nu) =$$
$$\rho_a\omega_x{}^a Q_\nu (g^{x\nu} - G^{pq}\alpha_p{}^x\alpha_q{}^\nu) = \rho_a\omega_x{}^a Q_\nu g^{x\nu} = \eta_x Q^x$$

This establishes that the second condition of (3) and the third condition of (4) are equivalent. Transforming the first condition of (3) as it was done in [1], we have

$$(\nabla_\nu\eta_x + \nabla_x\eta_\nu)\alpha_c{}^x\alpha_a{}^\nu + \eta_x(\nabla_c\alpha_a{}^x + \nabla_a\alpha_c{}^x) = 0$$

Substituting $\eta_x = \rho_b\omega_x{}^b$ in the above expression and using the fact that $\omega_x{}^b\nabla_a\alpha_c{}^x = 0$ [1], we obtain

$$(\nabla_\nu\eta_x + \nabla_x\eta_\nu)\alpha_a{}^x\alpha_c{}^\nu = 0$$

This constitutes the proof of the following theorem. If $\xi_x \dot{q}^x = C$ is a linear integral of a nonholonomic scleronomous system, then infinite number of vectors $\eta_x = \xi_x + \rho_p \times (q^x)\omega_x{}^p$ exist containing its generators $\rho_p$. However, of all these vectors, exactly one vector exists, $\eta_x = \rho_a\omega_x{}^a$ which lies in the admissible space. The necessary and sufficient conditions for the relation $\eta_x\dot{q}^x = C$ (where $\eta_x$ lies in the admissible space) to be the first linear integral of a nonholonomic system, are the following:

$$(\nabla_\nu\eta_x + \nabla_x\eta_\nu)\alpha_a{}^x\alpha_c{}^\nu = 0, \quad \eta^x\omega_x{}^p = 0, \quad \eta_x Q^x = 0 \tag{5}$$

Let us consider the first condition of (4) and the corresponding first condition of (5). The equations of the constraint-free system in inertial motion are as follows:

$$\delta\dot{q}^x / dt = 0 \tag{6}$$

Differentiating the left-hand side of the linear integral and using the last relations of (4) and (6), we obtain

$$\frac{\delta}{dt}(\eta_x\dot{q}^x) = \frac{\delta\eta_x}{dt}\dot{q}^x = \nabla_\nu\eta_x\dot{q}^\nu\dot{q}^x = \frac{1}{2}(\nabla_\nu\eta_x + \nabla_x\eta_\nu)\dot{q}^x\dot{q}^\nu = 0$$

We find that the first condition of (5) is equivalent to the following statement: the derivative of the left-hand side of the linear integral vanishes identically by virtue of the equations (6).

The first condition of (5) yields

$$\nabla_\nu\eta_x + \nabla_x\eta_\nu = 2\rho_{\lambda p}\omega_x{}^\lambda\omega_\nu{}^p + 2\rho_{p\lambda}\omega_x{}^p\omega_\nu{}^\lambda \tag{7}$$

On the other hand, differentiating the left-hand side of the linear integral and using (6) and (7), we obtain

$$\frac{\delta}{dt}(\eta_x\dot{q}^x) = \frac{1}{2}(\nabla_\nu\eta_x + \nabla_x\eta_\nu)\dot{q}^x\dot{q}^\nu = A_p\omega_\nu{}^p\dot{q}^\nu$$
$$(A_p = \rho_{\lambda p}\omega_x{}^\lambda\dot{q}^x + \rho_{p\lambda}\omega_x{}^\lambda\dot{q}^x)$$

Thus the first condition of (5) is equivalent to the following statement: the derivative of the left-hand side of the linear integral is, by virtue of (6), a linear combination of the equations of constraint.

Example. Let us consider the motion of Chaplygin sleigh on a horizontal plane [6] in the case when the direction of the runner is perpendicular to the segment connecting the center of gravity with the cutting point, and $M = 1$. The expression for the kinetic energy and the equation of nonholonomic constraint have the form

$$2T = (\xi^{\cdot} - \cos\varphi\,\varphi^{\cdot})^2 + (\eta^{\cdot} - \beta\sin\varphi\,\varphi^{\cdot})^2 + k_c^2\,\varphi^{\cdot 2}, \quad \eta^{\cdot} = \operatorname{tg}\varphi\,\xi$$

For the vectors $\alpha_\nu$ and $\omega^\nu$ we have

$$\alpha_1(1,\ 0,\ 0), \qquad \omega^1(k^2 + l^2,\ l\cos\varphi,\ l\sin\varphi)$$
$$\alpha_2(0,\ 1,\ \operatorname{tg}\varphi), \qquad \omega^2(l / \cos\varphi,\ 1,\ \operatorname{tg}\varphi)$$
$$\alpha_3(0,\ -\operatorname{tg}\varphi,\ 1), \qquad \omega^3(0,\ -\operatorname{tg}\varphi,\ 1)$$

The equations of motion of the nonholonomic system are as follows:

$$\varphi^{\cdot\cdot} = 0, \qquad \zeta^{\cdot\cdot} = -\operatorname{tg}\varphi\xi^{\cdot}\,\varphi^{\cdot}$$

It can easily be checked that $\check{\xi}^{\cdot} / \cos\varphi = C$ is the first linear integral of the nonholonomic system. The latter integral does not satisfy the second condition of (4). Setting $q^1 = \varphi$, $q^2 = \xi$ and $q^3 = \eta$ we obtain $\xi_1 = 0$, $\xi_2 = 1 / \cos\varphi$ and $\xi_3 = 0$ , hence

$$\xi_\varkappa \alpha_3{}^\varkappa = -\sin\varphi / \cos^2\varphi \neq 0$$

The vector $\xi_\varkappa$ can be represented by a linear combination of the vectors $\omega_\varkappa{}^\nu$ as follows:

$$\xi_\varkappa = -\frac{l}{k^2}\,\omega_\varkappa{}^1 + \left(1 + \frac{l^2}{k^2}\right)\cos\varphi\omega_\varkappa{}^2 - \sin\varphi\omega_\varkappa{}^3$$

The linear integral in question has a unique generating vector which lies in the admissible space and is

$$\eta_\varkappa = -\frac{l}{k^2}\,\omega_\varkappa{}^1 + \left(1 + \frac{l^2}{k^2}\right)\cos\varphi\omega_\varkappa{}^2$$

Substituting $\omega_\varkappa{}^1$ and $\omega_\varkappa{}^2$, we obtain $\eta_1 = 0$, $\eta_2 = \cos\varphi$ and $\eta_3 = \sin\varphi$.
The relation $\cos\varphi\xi^{\cdot} + \sin\varphi\eta^{\cdot} = C$ represents another form of the same linear integral. We shall show that the first condition of (5) obtained in the present paper holds for this integral, but not the first condition of (4) in [5]. In fact, using the constraint-free system in inertial motion

$$\varphi^{\cdot\cdot} = 0, \quad \xi^{\cdot\cdot} = -\beta\sin\varphi\varphi^{\cdot 2}, \quad \eta^{\cdot\cdot} = \beta\cos\varphi\varphi^{\cdot 2} \tag{8}$$

and differentiating the left-hand side of the linear integral we have, by virtue of (8),

$$\delta / dt(\xi^{\cdot}\cos\varphi + \eta^{\cdot}\sin\varphi) = \xi^{\cdot\cdot}\cos\varphi + \eta^{\cdot}\sin\varphi - \xi^{\cdot}\varphi^{\cdot}\sin\varphi +$$
$$\eta^{\cdot}\varphi^{\cdot}\cos\varphi = \varphi^{\cdot}\cos\varphi\,(\eta^{\cdot} - \xi^{\cdot}\operatorname{tg}\varphi)$$

The linear integrals of the nonholonomic system which satisfy conditions (4) are also linear integrals of the constraint-free system. The first of conditions (4) does not hold for those linear integrals of the nonholonomic system which are not simultaneously linear integrals of the constraint-free system, while the first of conditions (5) is satisfied for the generating vector $\eta_\varkappa = \rho_a\omega_\varkappa{}^a$ in the admissible space.

Conditions analogous to (5) can be obtained for any first integrals of the type shown in Sects. 2 and 3 of [1].

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# MULTIFREQUENCY RESONANCE OSCILLATIONS UNDER EXTERNAL PERTURBANCES

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Multifrequency oscillations in systems with a large number of degrees of freedom were considered in [1, 2]. In the present paper we study multifrequency oscillations of systems of a more specific form; we reduce the problem to the study of canonical systems of differential equations describing the resonance phenomena.

1. We consider a conservative system with $n$ degrees of freedom, which has a stable position of equilibrium; in a neighborhood of this position the system performs relatively small oscillations. The system is acted on by $N$ perturbations, which neither change the position of equilibrium nor lead out the motion of the system beyond the neighborhood of this position. We shall regard these perturbations as generalized coordinates (with index larger than $n$), which are specified functions of time. These coordinates enter formally into the expressions for the kinetic and potential energies (i.e. we assume that the conditional system with $n+N$ coordinates is a conservative system). We assume also that owing to a specified internal symmetry in the system, the expressions for the kinetic and potential energies are symmetric with respect to all of the $n+N$ generalized coordinates. Then

$$T=\frac{1}{2}\sum_{i,k=1}^{n+N} A_{ik}q_i^{\cdot}q_k^{\cdot}, \quad A_{ik}=a_{ik}+\frac{1}{2}\sum_{j,s=1}^{n+N} a_{ik}^{(js)}q_jq_s+\ldots \tag{1.1}$$

$$\Pi=\frac{1}{2}\sum_{i,k=1}^{n+N} c_{ik}q_iq_k+\frac{1}{24}\sum_{i,k,j,s=1}^{n+N} c_{ik}^{(js)}q_iq_kq_jq_s+\ldots \quad (c_{ik}=c_{ki}, c_{ik}^{(js)}=c_{jk}^{(is)}=\ldots,\ldots)$$

We assume that the symmetry of the coefficients in the expressions for $\Pi$, shown in the parentheses, holds also for the coefficients in the expression for $T$, i.e.

$$a_{ik}=a_{ki}, \qquad a_{ik}^{(js)}=a_{jk}^{(is)}=\ldots,\ldots \tag{1.2}$$

This assumption, without restricting the generality of results, leads to more simple and symmetric relationships.

We obtain the differential equations of motion from the equations (1.1) upon using the relations (1.2) and the fact that the $q_i$ are known functions of time for $i=n+1,\ldots,n+N$.

Let us assume that all perturbations are harmonic with frequencies $p_j (j=1,2,\ldots,N)$.